

An unitary invariant of semi-bounded operator and its application to inverse problems

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Abstract

Let L_0 be a closed densely defined symmetric semi-bounded operator with nonzero defect indexes in a separable Hilbert space \mathcal{H} . With L_0 we associate a metric space Ω_{L_0} that is named a *wave spectrum* and constructed from trajectories $\{u(t)\}_{t \geq 0}$ of a dynamical system governed by the equation $u_{tt} + (L_0)^*u = 0$. The wave spectrum is introduced through a relevant von Neumann operator algebra associated with the system. Wave spectra of unitary equivalent operators are isometric.

In inverse problems on *unknown* manifolds, one needs to recover a Riemannian manifold Ω via dynamical or spectral boundary data. We show that for a generic class of manifolds, Ω is *isometric* to the wave spectrum Ω_{L_0} of the minimal Laplacian $L_0 = -\Delta|_{C_0^\infty(\Omega \setminus \partial\Omega)}$ acting in $\mathcal{H} = L_2(\Omega)$, whereas L_0 is determined by the inverse data up to unitary equivalence. By this, one can recover the manifold by the scheme "the data $\Rightarrow L_0 \Rightarrow \Omega_{L_0} \stackrel{\text{isom}}{=} \Omega$ ".

The wave spectrum is relevant to a wide class of dynamical systems, which describe the finite speed wave propagation processes. The paper elucidates the operator background of the boundary control method (Belishev, 1986) based on relations of inverse problems to system and control theory.

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0 Introduction

0.1 Motivation

The paper introduces the notion of a *wave spectrum* of a symmetric semi-bounded operator in a Hilbert space. The impact comes from inverse problems of mathematical physics; the following is one of the motivating questions.

Let Ω be a smooth compact Riemannian manifold with the boundary Γ , $-\Delta$ the (scalar) Laplace operator, $L_0 = -\Delta|_{C_0^\infty(\Omega \setminus \Gamma)}$ the *minimal Laplacian* in $\mathcal{H} = L_2(\Omega)$. Assume that we are given with a unitary copy $\tilde{L}_0 = UL_0U^*$ in $\tilde{\mathcal{H}} = U\mathcal{H}$ (but U is unknown!). To what extent does \tilde{L}_0 determine the manifold Ω ?

So, we have no points, boundaries, tensors, etc, whereas the only thing given is an operator \tilde{L}_0 in a Hilbert space $\tilde{\mathcal{H}}$. Provided the operator is unitarily equivalent to L_0 , is it possible to extract Ω from \tilde{L}_0 ? Such a question is an "invariant" version of various setups of dynamical and spectral inverse problems on manifolds [2], [4].

0.2 Content

Substantially, the answer is affirmative: for a generic class of manifolds, any unitary copy of the minimal Laplacian determines Ω up to isometry (Theorem 1). A wave spectrum is a construction that realizes the determination $\tilde{L}_0 \Rightarrow \Omega$ and, thus, solves inverse problems. In more detail,

- With a closed densely defined symmetric semi-bounded operator L_0 of nonzero defect indexes in a separable Hilbert space \mathcal{H} we associate a metric space Ω_{L_0} (its wave spectrum). The space consists of the so-called eikonal operators (*eikonals*), so that Ω_{L_0} is a subset of the bounded operators algebra $\mathfrak{B}(\mathcal{H})$, whereas the metric on Ω_{L_0} is $\|\tau - \tau'\|_{\mathfrak{B}(\mathcal{H})}$.

The eikonals are constructed from the projections on the reachable sets of an abstract *dynamical system with boundary control* governed by the evolutionary equation $u_{tt} + L_0^*u = 0$. More precisely, they appear in the framework of a von Neumann algebra \mathfrak{N}_{L_0} associated with the system, whereas $\Omega_{L_0} \subset \mathfrak{N}_{L_0}$ is a set of the so-called *maximal eikonals*. The

peculiarity is that this algebra is endowed with an additional operation that we call a *space extension*.

Since the definition of Ω_{L_0} is of invariant character, the spectra Ω_{L_0} and $\Omega_{\tilde{L}_0}$ of the unitarily equivalent operators L_0 and \tilde{L}_0 turn out to be isometric (as metric spaces). So, a wave spectrum is a (hopefully, new) unitary invariant of a symmetric semi-bounded operator.

- A wide generic class of the so-called *simple manifolds* is introduced¹. The central Theorem 1 establishes that for a simple Ω , the wave spectrum of its minimal Laplacian L_0 is isometric to Ω . Hence, any unitary copy \tilde{L}_0 of L_0 determines the simple Ω up to isometry by the scheme $\tilde{L}_0 \Rightarrow \Omega_{\tilde{L}_0} \stackrel{\text{isom}}{=} \Omega_{L_0} \stackrel{\text{isom}}{=} \Omega$. In applications, it is the procedure, which recovers manifolds by the BC-method [2], [4]: the concrete inverse data determine a relevant \tilde{L}_0 , what enables one to realize the scheme.
- We discuss one more option: once the wave spectrum of the copy \tilde{L}_0 is found, the BC-procedure realizes elements of the space $\tilde{\mathcal{H}}$ as functions on $\Omega_{\tilde{L}_0}$ ². Thereafter, one can construct a *functional model* L_0^{mod} of the original Laplacian L_0 , the model being an operator in $\mathcal{H}^{\text{mod}} = L_{2,\mu}(\Omega_{\tilde{L}_0})$ related with L_0 through a similarity (gauge transform). Hopefully, this observation can be driven to a functional model of a class of symmetric semi-bounded operators. Presumably, this model will be *local*, i.e., satisfying $\text{supp } L_0^{\text{mod}} y \subseteq \text{supp } y$.

0.3 Comments

The concept of wave spectrum summarizes rich "experimental material" accumulated in inverse problems in the framework of the BC-method, and elucidates operator background of the latter. In fact, for the first time Ω_{L_0} has appeared in [1] in connection with the M.Kac problem; its later version (called a wave model) is presented in [4] (sec. 2.3.4). Owing to its invariant nature, Ω_{L_0} promises to be useful for further applications to unsolved inverse problems of elasticity theory, electrodynamics, graphs, etc.

Actually, a wave spectrum is an attribute not of a single operator but an algebra with space extension. In the scalar problems on manifolds, this

¹Roughly speaking, the simplicity means that the symmetry group of Ω is trivial.

²In the BC-method, such an option is interpreted as *visualization of waves* [4].

algebra is commutative, whereas its wave spectrum is identical to Gelfand's spectrum of the norm-closed subalgebra generated by eikonals. However, it is not clear whether this fact is of general character. The algebras that appear in the above mentioned unsolved problems, are *noncommutative* and the relation between their wave and Jacobson's spectra is not understood yet.

By the recent trend in the BC-method, to recover unknown manifolds via boundary inverse data is to find spectra of relevant algebras determined by the data [5]. We hope for further promotion of this approach.

1 Wave spectrum

1.1 Algebra with space extension

Let \mathcal{H} be a separable Hilbert space, $\text{Lat}\mathcal{H}$ the lattice of its (closed) subspaces; by $P_{\mathcal{A}}$ we denote the (orthogonal) projection onto $\mathcal{A} \in \text{Lat}\mathcal{H}$. Also, if \mathcal{A} is a non-closed lineal set, we put $P_{\mathcal{A}} := P_{\text{clos}\mathcal{A}}$. By $\mathfrak{B}(\mathcal{H})$ the bounded operator algebra is denoted.

An one-parameter family $E = \{E^t\}_{t \geq 0}$ of the maps $E^t : \text{Lat}\mathcal{H} \rightarrow \text{Lat}\mathcal{H}$ is said to be a *space extension* if

1. $E^0 = \text{id}$
2. $E^t\{0\} = \{0\}, \quad t \geq 0$
3. $t \leq t'$ and $\mathcal{A} \subseteq \mathcal{A}'$ imply $E^t\mathcal{A} \subseteq E^{t'}\mathcal{A}'$.

It is also convenient to regard E^t as an operation, which extends the projections, and write $E^t P_{\mathcal{A}} = P_{\mathcal{A}}^t := P_{E^t\mathcal{A}}$.

Assume that an extension E is given; let $a \subset \text{Lat}\mathcal{H}$ be a family of subspaces. By $\mathfrak{N}[E, a]$ we denote the minimal von Neumann operator algebra³, which contains all projections $\{P_{\mathcal{A}} \mid \mathcal{A} \in a\}$ and is closed with respect to E , i.e., $P \in \mathfrak{N}[E, a]$ implies $P^t = E^t P \in \mathfrak{N}[E, a]$, $t > 0$. As is easy to see, such an algebra is well defined. As any von Neumann algebra, $\mathfrak{N}[E, a]$ is determined by the set $\text{Proj } \mathfrak{N}[E, a]$ of its projections, whereas its additional property is

$$E^t \text{Proj } \mathfrak{N}[E, a] \subset \text{Proj } \mathfrak{N}[E, a], \quad t > 0.$$

³i.e., a unital weakly closed self-adjoint subalgebra of $\mathfrak{B}(\mathcal{H})$: see [11]

Fix a projection $P \in \mathfrak{N}[E, a]$; a positive self-adjoint operator in \mathcal{H} of the form

$$\tau_P := \int_0^\infty t dP^t,$$

where $P^t = E^t P$, is said to be an *eikonal*⁴; the set of eikonals is denoted by $\text{Eik } \mathfrak{N}[E, a]$.

Let us say that we deal with the *bounded case* if each eikonal is a bounded operator (and, hence, belongs to $\mathfrak{N}[E, a]$) and the set $\text{Eik } \mathfrak{N}[E, a]$ is bounded in $\mathfrak{B}(\mathcal{H})$:

$$\sup \{ \|\tau\| \mid \tau \in \text{Eik } \mathfrak{N}[E, a] \} < \infty. \quad (1.1)$$

Otherwise, the situation is referred to as *unbounded case* (see the end of sec 3.4).

Convention 1 *Unless otherwise specified, we deal with the bounded case.*

Recall that the set of self-adjoint operators is partially ordered: for $A, B \in \mathfrak{B}(\mathcal{H})$ the relation $A \leq B$ means $(Ax, x) \leq (Bx, x)$, $x \in \mathcal{H}$. Any monotonic bounded sequence $A_1 \leq A_2 \leq \dots$, $\sup \|A_j\| < \infty$ converges in the strong operator topology to $s\text{-}\lim A_j \in \mathfrak{B}(\mathcal{H})$ (see, e.g., [6]).

Extend the name *eikonal* to all elements of the set $s\text{-}\text{clos } \text{Eik } \mathfrak{N}[E, a] \subset \mathfrak{N}[E, a]$ and denote the extended set by the same symbol $\text{Eik } \mathfrak{N}[E, a]$. An eikonal τ is said to be *maximal* if $\tau \geq \tau'$ for any eikonal τ' comparable with τ . Let $\Omega_{\mathfrak{N}[E, a]} \subset \text{Eik } \mathfrak{N}[E, a]$ be the *set of maximal eikonals*.

Lemma 1 *The set $\Omega_{\mathfrak{N}[E, a]}$ is nonempty.*

Proof By (1.1), any totally ordered family of eikonals $\{\tau_\alpha\}$ has an upper bound $s\text{-}\lim \tau_\alpha$, which is also an eikonal. Hence, the Zorn lemma implies $\Omega_{\mathfrak{N}[E, a]} \neq \emptyset$. \square

The $\mathfrak{B}(\mathcal{H})$ -norm induces the distance $\|\tau - \tau'\|$ in $\Omega_{\mathfrak{N}[E, a]}$ and makes it into a metric space, which is determined by the extension E and the initial reserve of subspaces a . The space $\Omega_{\mathfrak{N}[E, a]}$ is the main subject of our paper.

⁴the term is taken from the motivating applications

1.2 Extension E_L

Here we introduce the space extension associated with a semi-bounded self-adjoint operator. Without lack of generality, it is assumed positive definite: let

$$L = L^* = \int_0^\infty \lambda dQ_\lambda; \quad (Ly, y) \geq \varkappa \|y\|^2, \quad y \in \text{Dom } L \subset \mathcal{H}, \quad (1.2)$$

where dQ_λ is the spectral measure of L and \varkappa is a positive constant. Such an operator governs the evolution of a dynamical system

$$v_{tt} + Lv = h, \quad t > 0 \quad (1.3)$$

$$v|_{t=0} = v_t|_{t=0} = 0, \quad (1.4)$$

where $h \in L_2^{\text{loc}}((0, \infty); \mathcal{H})$ is a \mathcal{H} -valued function of time (*control*). Its finite energy class solution $v = v^h(t)$ is represented by the Duhamel formula

$$\begin{aligned} v^h(t) &= \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)L^{\frac{1}{2}} \right] h(s) ds = \langle \text{see(1.2)} \rangle \\ &= \int_0^t ds \int_0^\infty \frac{\sin \sqrt{\lambda}(t-s)}{\sqrt{\lambda}} dQ_\lambda h(s), \quad t \geq 0 \end{aligned} \quad (1.5)$$

(see, e.g., [6]). In system theory, $v^h(\cdot)$ is referred to as a *trajectory*; $v^h(t) \in \mathcal{H}$ is a *state* at the moment t . In applications, v^h describes a *wave* initiated by a source h . Note that in the case of $\varkappa \leq 0$ the problem (1.3), (1.4) is also well defined but the representation (1.5) is of slightly more complicated form. Thus, the assumption $\varkappa > 0$ is accepted just for simplicity.

Fix a subspace $\mathcal{A} \subset \mathcal{H}$; the set

$$\mathcal{V}_{\mathcal{A}}^t := \{v^h(t) \mid h \in L_2^{\text{loc}}((0, \infty); \mathcal{A})\}, \quad t > 0$$

of all states produced by \mathcal{A} -valued controls is called *reachable* (at the moment t). Reachable sets increase as \mathcal{A} increases and/or t grows. Indeed, the representation (1.5) easily implies

$$v^{\mathcal{T}_\xi h}(t) = (\mathcal{T}_\xi v^h)(t), \quad t \geq 0, \quad (1.6)$$

where \mathcal{T}_ξ is the right shift operator in $L_2^{\text{loc}}((0, \infty); \mathcal{H})$:

$$(\mathcal{T}_\xi g)(t) := \begin{cases} 0, & 0 \leq t < \xi \\ g(t - \xi), & t \geq \xi \end{cases}$$

with $\xi \geq 0$. For $0 < t \leq t'$ and $\mathcal{A} \subseteq \mathcal{A}'$, we have

$$\mathcal{V}_{\mathcal{A}}^t \ni v^h(t) = (\mathcal{T}_{t'-t} v^h)(t') = \langle \text{see (1.6)} \rangle = v^{\mathcal{T}_{t'-t} h}(t') \in \mathcal{V}_{\mathcal{A}}^{t'} \subseteq \mathcal{V}_{\mathcal{A}'}^{t'},$$

i.e., the inclusion

$$\mathcal{V}_{\mathcal{A}}^t \subseteq \mathcal{V}_{\mathcal{A}'}^{t'}, \quad 0 < t \leq t' \quad (1.7)$$

does hold.

Define a family $E_L = \{E^t\}_{t \geq 0}$ of the maps $E^t : \text{Lat} \mathcal{H} \rightarrow \text{Lat} \mathcal{H}$ by

$$E^0 \mathcal{A} := \mathcal{A}, \quad E^t \mathcal{A} := \text{clos } \mathcal{V}_{\mathcal{A}}^t, \quad t > 0. \quad (1.8)$$

Lemma 2 E_L is a space extension.

Proof The properties 1 and 2 (see section 1.1) easily follow from the definitions and the obvious relation $\mathcal{V}_{\{0\}}^t = \{0\}$; the property 3 for $0 < t \leq t'$ is seen from (1.7). Thus, it remains to verify 3 for $t = 0$, i.e., to check that $0 = t \leq t'$ and $\mathcal{A} \subseteq \mathcal{A}'$ leads to $E^0 \mathcal{A} \subseteq E^{t'} \mathcal{A}'$ or, the same, that $\mathcal{A} \subseteq E^r \mathcal{A}$ for all $r > 0$ and $\mathcal{A} \neq \{0\}$.

Take a nonzero $y \in \mathcal{A}$ and consider (1.3), (1.4) with the control $h_\varepsilon(t) = \varphi_\varepsilon(t)y$, where

$$\varphi_\varepsilon(t) := \begin{cases} 0, & t \in [0, r - 2\varepsilon) \\ \frac{1}{\varepsilon^2}, & t \in [r - 2\varepsilon, r - \varepsilon) \\ -\frac{1}{\varepsilon^2}, & t \in [r - \varepsilon, r) \\ 0, & t \in [r, \infty) \end{cases}$$

($\varepsilon > 0$ is small); note that

$$\int_0^r \varphi_\varepsilon(t) f(t) dt \xrightarrow{\varepsilon \rightarrow 0} -f'(r)$$

for smooth f 's, i.e., $\varphi_\varepsilon(t)$ converges to $\delta'(t - r)$ as a distribution. Define

$$\psi_\varepsilon(\lambda) := \int_0^r \frac{\sin[\sqrt{\lambda}(r - t)]}{\sqrt{\lambda}} \varphi_\varepsilon(t) dt = \frac{2 \cos(\sqrt{\lambda} \varepsilon) - \cos(\sqrt{\lambda} 2\varepsilon) - 1}{\varepsilon^2 \lambda}$$

and note that $\psi_\varepsilon(\lambda) \xrightarrow{\varepsilon \rightarrow 0} 1$ uniformly w.r.t. λ in any compact segment $[\varkappa, N]$.

Therefore, one has

$$\begin{aligned} \|y - v^{h_\varepsilon}(r)\|^2 &= \langle \text{see (1.5)} \rangle = \left\| y - \int_0^r dt \int_{\mathcal{K}} \frac{\sin[\sqrt{\lambda}(r-t)]}{\sqrt{\lambda}} dQ_\lambda[\varphi_\varepsilon(t)y] \right\|^2 = \\ &= \left\| y - \int_{\mathcal{K}} \psi_\varepsilon(\lambda) dQ_\lambda y \right\|^2 = \left\| \int_{\mathcal{K}} [1 - \psi_\varepsilon(\lambda)] dQ_\lambda y \right\|^2 = \\ &= \int_{\mathcal{K}} |1 - \psi_\varepsilon(\lambda)|^2 d\|Q_\lambda y\|^2 \xrightarrow{\varepsilon \rightarrow 0} 0 \end{aligned}$$

by the properties of ψ_ε . The order of integration change is easily justified by the Fubini Theorem.

Thus, $y = \lim_{\varepsilon \rightarrow 0} v^{h_\varepsilon}(r)$, whereas $v^{h_\varepsilon}(r) \in E^r \mathcal{A}$ holds. By the closeness of $E^r \mathcal{A}$, we get $y \in E^r \mathcal{A}$. Hence, $\mathcal{A} \subseteq E^r \mathcal{A}$. \square

So, with each positive definite operator L one associates the certain space extension E_L by (1.8).

1.3 Algebras $\mathfrak{N}_{L,\mathcal{D}}$ and \mathfrak{N}_{L_0}

Return to the system (1.3)–(1.4) and fix a nonzero subspace $\mathcal{D} \in \text{Lat} \mathcal{H}$ that we'll call a *directional subspace*. It determines a class

$$\mathcal{M}_{\mathcal{D}} := \{h \in C^\infty([0, \infty); \mathcal{D}) \mid \text{supp } h \subset (0, \infty)\} \quad (1.9)$$

of smooth \mathcal{D} -valued controls vanishing near $s = 0$, and the sets

$$\begin{aligned} \mathcal{U}^t &:= \left\{ h(t) - v^{h''}(t) \mid h \in \mathcal{M}_{\mathcal{D}} \right\} = \langle \text{see (1.5)} \rangle = \\ &= \left\{ h(t) - \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)L^{\frac{1}{2}} \right] h''(s) ds \mid h \in \mathcal{M}_{\mathcal{D}} \right\}, \quad t \geq 0 \end{aligned} \quad (1.10)$$

(here $(\cdot)' := \frac{d}{ds}$), which we also call *reachable*⁵. As can be easily derived from (1.6), the sets \mathcal{U}^t increase as t grows.

Now, take

$$a_{L,\mathcal{D}} := \{\text{clos } \mathcal{U}^t\}_{t \geq 0} \subset \text{Lat} \mathcal{H}$$

⁵The meaning of this definition and term is clarified later on, when we deal with systems with boundary control in sec 2.2

in capacity of the initial family of subspaces (see sec 1.1). The pair L, \mathcal{D} determines the algebra with extension

$$\mathfrak{N}_{L, \mathcal{D}} := \mathfrak{N}[E_L, a_{L, \mathcal{D}}]$$

and its eikonals $\text{Eik } \mathfrak{N}_{L, \mathcal{D}}$. Assuming that (1.1) holds for $\mathfrak{N}_{L, \mathcal{D}}$, the set of maximal eikonals

$$\Omega_{L, \mathcal{D}} := \Omega_{\mathfrak{N}_{L, \mathcal{D}}}$$

is well defined and called a *wave spectrum* of the pair L, \mathcal{D} . The operator

$$\tau^\partial := \int_0^\infty t dP_{\mathcal{U}^t} \quad (1.11)$$

is said to be a *boundary eikonal*. The set

$$\partial\Omega_{L, \mathcal{D}} := \{\tau \in \Omega_{L, \mathcal{D}} \mid \tau \geq \tau^\partial\} \subset \Omega_{L, \mathcal{D}} \quad (1.12)$$

is a *boundary* of the wave spectrum.

Thus, each pair L, \mathcal{D} determines an algebra with space extension $\mathfrak{N}_{L, \mathcal{D}}$ and all corresponding attributes.

Let L_0 be a closed densely defined symmetric semi-bounded operator with *nonzero* defect indexes $n_\pm = n \leq \infty$. As is easy to see, such an operator is necessarily unbounded. For the sake of simplicity, it is assumed positive definite: $(L_0 y, y) \geq \varkappa \|y\|^2$, $y \in \text{Dom } L_0$ with $\varkappa > 0$. Let L be the extension of L_0 by Friedrichs, so that $L = L^* \geq \varkappa \mathbb{I}$ and

$$L_0 \subset L \subset L_0^* \quad (1.13)$$

holds [6]. Also, note that $1 \leq \dim \text{Ker } L_0^* = n \leq \infty$. Taking

$$\mathcal{D} := \text{Ker } L_0^*$$

as a directional subspace, we can constitute the pair $L, \text{Ker } L_0^*$, which determines the algebra

$$\mathfrak{N}_{L_0} := \mathfrak{N}_{L, \text{Ker } L_0^*}$$

and its eikonals $\text{Eik } \mathfrak{N}_{L_0}$. If (1.1) holds, the set of maximal eikonals

$$\Omega_{L_0} := \Omega_{L, \text{Ker } L_0^*}$$

is well defined and referred to as a *wave spectrum* of the operator L_0 ; its subset $\partial\Omega_{L_0}$ is a *boundary* of the wave spectrum.

So, with every L_0 of the above-mentioned class, one associates the algebra \mathfrak{N}_{L_0} and the algebra eikonals $\text{Eik } \mathfrak{N}_{L_0}$. If the latter set is bounded, the operator L_0 possesses the wave spectrum $\Omega_{L_0} \neq \emptyset$ ⁶.

2 DSBC

2.1 Green system

Dynamical systems with boundary control (DSBC) are defined in the next sections 2.2 and 2.3; here we introduce a basic ingredient of the definition. The ingredient is a collection $\{\mathcal{H}, \mathcal{G}; A, \Gamma_0, \Gamma_1\}$ of the separable Hilbert spaces \mathcal{H} and \mathcal{G} , and the densely defined operators $A : \mathcal{H} \rightarrow \mathcal{H}$, $\Gamma_k : \mathcal{H} \rightarrow \mathcal{G}$ ($k = 0, 1$) connected through the Green formula

$$(Au, v)_{\mathcal{H}} - (u, Av)_{\mathcal{H}} = (\Gamma_0 u, \Gamma_1 v)_{\mathcal{G}} - (\Gamma_1 u, \Gamma_0 v)_{\mathcal{G}}$$

(see [10]). Such a collection is said to be a *Green system*; \mathcal{G} and Γ_k are referred to as a *boundary values space* (BVS) and the *boundary operators* respectively. In the applications, which we deal with later on, the following is also provided:

1. $\text{Dom } \Gamma_k \supseteq \text{Dom } A$ holds, whereas A is such that the restriction

$$A|_{\text{Ker } \Gamma_0 \cap \text{Ker } \Gamma_1} =: L_0$$

is a densely defined symmetric positive definite operator with nonzero defect indexes and $\overline{A} = L_0^*$ is valid ('bar' is the operator closure)

2. the restriction

$$A|_{\text{Ker } \Gamma_0} =: L$$

coincides with the Friedrichs extension of L_0 , so that we have

$$L_0 \subset L \subset L_0^* = \overline{A}, \quad (2.1)$$

whereas L^{-1} is bounded and defined on \mathcal{H}

⁶However, there are examples in applications, in which Ω_{L_0} consists of a single point.

3. for the subspaces $\mathcal{A} := \text{Ker } A$ and $\mathcal{D} := \text{Ker } L_0^*$, the relations

$$\text{clos } \mathcal{A} = \mathcal{D}, \quad \text{clos } \Gamma_0 \mathcal{A} = \mathcal{G} \quad (2.2)$$

are valid.

These properties are in consent with the Green system theory by V.A.Ryzhov, which puts them as the basic axioms. Note, that there are a few versions of such an axiomatics but the one proposed in [12] is most relevant for applications to the forward and inverse *multidimensional* problems of mathematical physics.

Convention 2 *In what follows we deal with the Green systems, which satisfy the conditions 1–3.*

As is shown in [12], the axioms provide the following:

- the map $\Pi := (\Gamma_1 L^{-1})^* : \mathcal{G} \rightarrow \mathcal{H}$ is bounded, whereas $\text{Ran } \Pi$ is dense in \mathcal{D}
- the subspace \mathcal{A} admits the characterization

$$\mathcal{A} = \{y \in \text{Dom } A \mid \Pi \Gamma_0 y = y\} \quad (2.3)$$

- since L is the extension of L_0 by Friedrichs, we have

$$\text{Dom } L_0 = L^{-1}[\mathcal{H} \ominus \mathcal{D}], \quad L_0 = L|_{L^{-1}[\mathcal{H} \ominus \mathcal{D}]} \quad (2.4)$$

that easily follows from the definition of such an extension (see [6]).

Example Let Ω be a C^∞ -smooth compact Riemannian manifold with the boundary Γ , Δ the (scalar) Laplace operator in $\mathcal{H} := L_2(\Omega)$, ν the outward normal on Γ , $\mathcal{G} := L_2(\Gamma)$. Denote ⁷

$$A = -\Delta|_{H^2(\Omega)}, \quad \Gamma_0 := (\cdot)|_\Gamma, \quad \Gamma_1 := \frac{\partial}{\partial \nu}(\cdot)|_\Gamma,$$

so that $\Gamma_{0,1}$ are the trace operators. The collection $\{\mathcal{H}, \mathcal{G}; A, \Gamma_0, \Gamma_1\}$ is a Green system, whereas other operators, which appear in the framework of Ryzhov's axiomatics, are the following:

$$L_0 = -\Delta|_{H_0^2(\Omega)}$$

⁷ $H^k(\dots)$ are the Sobolev classes; $H_0^2(\Omega) = \{y \in H^2(\Omega) \mid y = |\nabla y| = 0 \text{ on } \Gamma\}$.

is the minimal Laplacian;

$$L = -\Delta|_{H^2(\Omega) \cap H_0^1(\Omega)}$$

is the self-adjoint Dirichlet Laplacian;

$$L_0^* = -\Delta|_{\{y \in \mathcal{H} \mid \Delta y \in \mathcal{H}\}}$$

is the maximal Laplacian;

$$\mathcal{A} = \{y \in H^2(\Omega) \mid \Delta y = 0\}$$

is the set of harmonic functions of the class $H^2(\Omega)$;

$$\mathcal{D} = \{y \in \mathcal{H} \mid \Delta y = 0\}$$

is the subspace of all harmonic functions in $L_2(\Omega)$; $\Pi : \mathcal{G} \rightarrow \mathcal{H}$ is the harmonic continuation operator (the Diriclet problem solver):

$$\Pi\varphi = u : \quad \Delta u = 0 \text{ in } \Omega, \quad u|_{\Gamma} = \varphi.$$

2.2 Evolutionary DSBC

Note in advance that the goal of the sections 2.2, 2.3 is not to obtain results but motivate the above introduced objects and notions. That is why the presentation ignores certain technical details.

Definition The Green system determines an evolutionary *dynamical system with boundary control* of the form

$$u_{tt} + A u = 0 \quad \text{in } \mathcal{H}, \quad 0 < t < \infty \quad (2.5)$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \mathcal{H} \quad (2.6)$$

$$\Gamma_0 u = f(t) \quad \text{in } \mathcal{G}, \quad 0 \leq t < \infty, \quad (2.7)$$

where f is a *boundary control* (\mathcal{G} -valued function of time), $u = u^f(t)$ is the solution (*wave*).

Assign f to a class \mathcal{F}_+ if it belongs to $C^\infty([0, \infty); \mathcal{G})$, takes the values in $\Gamma_0 \text{Dom } A$, and vanishes near $t = 0$, i.e., satisfies $\text{supp } f \subset (0, \infty)$. Also, note that $f \in \mathcal{F}_+$ implies $\Pi(f(\cdot)) \in \mathcal{M}_{\mathcal{D}}$, where $\mathcal{D} = \text{Ker } L_0^*$ and $\mathcal{M}_{\mathcal{D}}$ is defined by (1.9).

Lemma 3 For $f \in \mathcal{F}_+$, the classical solution u^f to problem (2.5)–(2.7) is represented in the form

$$u^f(t) = h(t) - \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)L^{\frac{1}{2}} \right] h''(s) ds, \quad t \geq 0 \quad (2.8)$$

with $h := \Pi(f(\cdot)) \in \mathcal{M}_{\mathcal{D}}$.

Proof Introducing a new unknown $w = w^f(t) := u^f(t) - \Pi(f(t))$ and taking into account (2.3), we easily get the system

$$\begin{aligned} w_{tt} + Aw &= -\Pi(f_{tt}(t)) && \text{in } \mathcal{H}, \quad 0 < t < \infty \\ w|_{t=0} = w_t|_{t=0} &= 0 && \text{in } \mathcal{H} \\ \Gamma_0 w &= 0 && \text{in } \mathcal{G}, \quad 0 \leq t < \infty. \end{aligned}$$

With regard to the definition of the operator L (see the axiom 2), this problem can be rewritten in the form

$$\begin{aligned} w_{tt} + Lw &= -h_{tt} && \text{in } \mathcal{H}, \quad 0 < t < \infty \\ w|_{t=0} = w_t|_{t=0} &= 0 && \text{in } \mathcal{H} \end{aligned}$$

and then solved by the Duhamel formula

$$w^f(t) = - \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)L^{\frac{1}{2}} \right] h''(s) ds.$$

Returning back to $u^f = w^f + \Pi f$, we arrive at (2.8). \square

The sets

$$\begin{aligned} \mathcal{U}_+^t &:= \{u^f(t) \mid f \in \mathcal{F}_+\} = \langle \text{see (2.8)} \rangle = \\ &\left\{ h(t) - \int_0^t L^{-\frac{1}{2}} \sin \left[(t-s)L^{\frac{1}{2}} \right] h''(s) ds \mid h = \Pi f(\cdot), f \in \mathcal{F}_+ \right\}, \quad t \geq 0 \end{aligned} \quad (2.9)$$

are said to be *reachable from boundary*. In the mean time, the Green system, which governs the DSBC, determines the certain pair L, \mathcal{D} , which in turn determines the family $\{\mathcal{U}^t\}$ by (1.10). Comparing the definitions, we easily conclude that the inclusion $\mathcal{U}_+^t \subset \mathcal{U}^t$ holds. Moreover, the density properties (2.2) enable one to derive

$$\text{clos } \mathcal{U}_+^t = \text{clos } \mathcal{U}^t, \quad t \geq 0 \quad (2.10)$$

and it is the relation, which inspires the definition (1.10) and motivates the use of the term "reachable set" for \mathcal{U}^t in the general case, where neither the boundary value space nor the boundary operators are defined.

Illustration Consider the Example at the end of sec 2.1. The DSBC (2.5)–(2.7) associated with the Riemannian manifold is governed by the wave equation and is of the form

$$u_{tt} - \Delta u = 0 \quad \text{in } \Omega \times (0, \infty) \quad (2.11)$$

$$u|_{t=0} = u_t|_{t=0} = 0 \quad \text{in } \Omega \quad (2.12)$$

$$u|_{\Gamma} = f(t) \quad \text{for } 0 \leq t < \infty \quad (2.13)$$

with a boundary control $f \in \mathcal{F} := L_2^{\text{loc}}((0, \infty); L_2(\Gamma))$; the solution $u = u^f(x, t)$ describes a wave, which is initiated by boundary sources and propagates from the boundary into the manifold with the speed 1. For $f \in \mathcal{F}_+ := C^\infty([0, \infty); C^\infty(\Gamma))$ provided $\text{supp } f \subset (0, \infty)$, the solution u^f is classical. By the finiteness of the wave propagation speed, at a moment t the waves fill the near-boundary subdomain

$$\Omega^t[\Gamma] := \{x \in \Omega \mid \text{dist}(x, \Gamma) < t\}.$$

Correspondingly, the reachable sets \mathcal{U}_+^t increase as t grows and the relation

$$\mathcal{U}_+^t \subset \mathcal{H}^t, \quad t \geq 0 \quad (2.14)$$

holds, where $\mathcal{H}^t := \text{clos} \{y \in \mathcal{H} \mid \text{supp } y \subset \Omega^t[\Gamma]\}$.

So, if the pair L, \mathcal{D} (or the operator L_0) appears in the framework of a Green system, then $\{\mathcal{U}^t\}$ introduced by the general definition (1.10) may be imagine as the sets of waves produced by boundary controls. The question arises, what is the meaning of the corresponding wave spectrum $\Omega_{L, \mathcal{D}}$ ($= \Omega_{L_0}$)? In a sense, it is the question, which this paper is written for. The answer (postponed till section 3) is that Ω_{L_0} is a wave guide body, in which such waves propagate.

Controllability Return to the abstract DSBC (2.5)–(2.7) and define its certain property. The definition is premised with the following observation. Since the class of controls \mathcal{F}_+ satisfies $\frac{d^2}{dt^2} \mathcal{F}_+ = \mathcal{F}_+$, the reachable sets (2.9) satisfy $A\mathcal{U}_+^t = \mathcal{U}_+^t$. Indeed, taking $f \in \mathcal{F}_+$ we have

$$Au^f(t) = \langle \text{see (2.5)} \rangle = -u_{tt}^f(t) = u^{-f''}(t) \in \mathcal{U}_+^t \quad (2.15)$$

and, by the same relations,

$$u^f(t) = Au^g(t)$$

with $g = -(\int_0^t)^2 f \in \mathcal{F}_+$. Hence, the sets \mathcal{U}_+^t reduce the operator A , so that its parts $A|_{\mathcal{U}_+^t}$ are well defined.

The DSBC (2.5)–(2.7) is said to be *controllable* from boundary for the time $t = T$ if the (operator) closure of the part $A|_{\mathcal{U}_+^T}$ coincides with \overline{A} , i.e., the relation

$$\text{clos } \{ \{u^f(T), Au^f(T)\} \mid f \in \mathcal{F}_+ \} = \text{clos graph } A \quad (2.16)$$

(the closure in $\mathcal{H} \times \mathcal{H}$) is valid, where

$$\text{graph } A := \{ \{y, Ay\} \mid y \in \text{Dom } A \} .$$

Controllability means two things. First, since A is densely defined in \mathcal{H} , (2.16) implies

$$\text{clos } \mathcal{U}_+^t = \mathcal{H}, \quad t \geq T,$$

i.e., for large times the reachable sets become rich enough: dense in \mathcal{H} . Second, the "wave part" $A|_{\mathcal{U}_+^T}$ of the operator A , which governs the evolution of the system, represents the operator in substantial: it coincides with A up to closure.

In applications to problems in bounded domains, such a property "ever holds" (for large enough times T). In particular, the system (2.11)–(2.13) is controllable for any $T > \max_{x \in \Omega} \text{dist}(x, \Gamma)$ (see [2], [4]).

Let us represent the property (2.16) in the form available for what follows. At first, with regard to (2.1) and (2.15), it can be written as

$$\text{clos } \left\{ \{u^f(T), u^{-f''}(T)\} \mid f \in \mathcal{F}_+ \right\} = \text{graph } L_0^*. \quad (2.17)$$

Further, for each $t \geq 0$, introduce a *control operator* ("input \rightarrow state" map) $W^t : \mathcal{F} \rightarrow \mathcal{H}$, $\text{Dom } W^t = \mathcal{F}_+$,

$$W^t f := u^f(t).$$

In terms of this map, (2.17) takes the form

$$\text{clos } \left\{ \{W^T f, W^T(-f'')\} \mid f \in \mathcal{F}_+ \right\} = \text{graph } L_0^*. \quad (2.18)$$

The control operator can be regarded as an operator from a Hilbert space $\mathcal{F}^t := L_2((0, t); \mathcal{G})$ to the space \mathcal{H} with $\text{Dom } W^t = \mathcal{F}_+^t := \{f|_{[0, t]} \mid f \in \mathcal{F}_+\}$. As such, it can be represented in the form of the *polar decomposition*

$$W^t = U^t |W^t|,$$

where $|W^t| := ((W^t)^* W^t)^{\frac{1}{2}}$ and U^t is an isometry from $\text{clos Ran } |W^t| \subset \mathcal{F}^t$ onto $\text{clos Ran } W^t \subset \mathcal{H}$ (see, e.g., [6]). For $t = T$, one has $\text{clos Ran } W^T = \mathcal{H}$, so that U^T is a unitary operator from the (sub)space $\tilde{\mathcal{H}} := \text{clos Ran } |W^T|$ onto \mathcal{H} . Correspondingly, the operator

$$\tilde{L}_0^* := (U^T)^* L_0^* U^T$$

acting in $\tilde{\mathcal{H}}$ turns out to be unitarily equivalent to L_0^* . As result, (2.18) can be written in the final form

$$\text{clos} \left\{ \{|W^T|f, |W^T|(-f'')\} \mid f \in \mathcal{F}_+ \right\} = \text{graph } \tilde{L}_0^*. \quad (2.19)$$

As a consequence, we get

Proposition 1 *If the DSBC (2.5)–(2.7) is controllable for the time T then the operator $|W^T|$ determines the operator L_0^* up to unitary equivalence.*

Response operator In the DSBC (2.5)–(2.7), an "input→output" correspondence is described by the *response operator* $R : \mathcal{F} \rightarrow \mathcal{F}$, $\text{Dom } R = \mathcal{F}_+$,

$$(Rf)(t) := \Gamma_1(u^f(t)), \quad t \geq 0.$$

Also, the reduced operators

$$R^t f := (Rf)|_{[0, t]}$$

are in use and play the role of the data in dynamical inverse problems.

The key fact of the BC-method is that the operator R^{2t} determines the operator $(W^t)^* W^t$ through a simple and explicit relation: see [2], [3], [4]. Hence, R^{2t} determines the modulus $|W^t|$. Combining this fact with the Proposition 1, we arrive at

Proposition 2 *If the DSBC (2.5)–(2.7) is controllable for the time T then its response operator R^{2T} determines the operator L_0^* (and, hence, the operator $L_0 = L_0^{**}$ and its Friedrichs extension L) up to unitary equivalence.*

As illustration, the response operator of the DSBC (2.11)–(2.13) is

$$R^{2T} : f \mapsto \frac{\partial u^f}{\partial \nu} \Big|_{\Gamma \times [0, 2T]}.$$

By the aforesaid, given for a fixed $T > \max_{x \in \Omega} \text{dist}(x, \Gamma)$ this operator determines the operator L_0 up to a unitary equivalence.

2.3 Stationary DSBC

Our presentation follows the paper [12]. The basic object is the Green system $\{\mathcal{H}, \mathcal{G}; A, \Gamma_0, \Gamma_1\}$ and the associated operators L_0, L (see sec 2.1).

Definition Along with the evolutionary DSBC, one associates with the Green system the problem

$$(A - z\mathbb{I})u = 0 \quad \text{in } \mathcal{H}, \quad z \in \mathbb{C} \quad (2.20)$$

$$\Gamma_0 u = \varphi \quad \text{in } \mathcal{G} \quad (2.21)$$

that is referred to as a *stationary* DSBC. For $\varphi \in \Gamma_0 \text{Dom } A$ and $z \in \mathbb{C} \setminus \text{spec } L$, such a problem has a unique solution $u = u_z^\varphi$, which is a $\text{Dom } A$ -valued function of z .

Weyl function The "input \rightarrow output" correspondence in the system (2.20)–(2.21) is realized by an operator-valued function

$$M(z)\varphi := \Gamma_1 u_z^\varphi, \quad z \notin \text{spec } L$$

that is called *Weyl function* and plays the role of the data in frequency domain inverse problems.

The following fact proven in [12] is of crucial value. Recall that a symmetric operator in \mathcal{H} is said to be *completely non-selfadjoint* if there is no subspace in \mathcal{H} , in which the operator induces a self-adjoint part.

Proposition 3 *If the Green system, which determines the DSBC (2.20)–(2.21), is such that the operator L_0 is completely non-selfadjoint, then the Weyl function of the DSBC determines the operator L_0 up to unitary equivalence.*

Illustration Consider the Example at the end of sec 2.1. The DSBC (2.20)–(2.21) associated with the Riemannian manifold is

$$(A + z)u = 0 \quad \text{in } \Omega \quad (2.22)$$

$$u|_{\Gamma} = \varphi, \quad (2.23)$$

where $A = -\Delta|_{H^2(\Omega)}$. The operator $L_0 = -\Delta|_{H_0^2(\Omega)}$ is completely non-selfadjoint. Indeed, otherwise there exists a subspace $\mathcal{K} \subset \mathcal{H}$ such that the operator $L_0^{\mathcal{K}} := -\Delta|_{\mathcal{K} \cap H_0^2(\Omega)} \neq \mathbb{O}$ is self-adjoint in \mathcal{K} . In the mean time, $L_0^{\mathcal{K}}$ is a part of L , the latter being a self-adjoint operator with the discrete spectrum. Hence, $\text{spec } L_0^{\mathcal{K}}$ is also purely discrete; each of its eigenfunctions satisfies $-\Delta\phi = \lambda\phi$ in Ω and belongs to $H_0^2(\Omega)$. The latter implies $\phi = \frac{\partial\phi}{\partial\nu} = 0$ on Γ , which leads to $\phi \equiv 0$ by the well-known E.Landis uniqueness theorem for solutions to the Cauchy problem for elliptic equations. Hence, $L_0^{\mathcal{K}} = \mathbb{O}$ in contradiction to the assumptions.

The Weyl function of the system is

$$M(z)\varphi = \left. \frac{\partial u_z^{\varphi}}{\partial\nu} \right|_{\Gamma} \quad (z \notin \text{spec } L).$$

By the aforesaid, the function M determines the operator L_0 up to a unitary equivalence.

Besides the Weyl function, there is one more kind of inverse data associated with to the DSBC (2.22)–(2.23). Let $\{\lambda_k\}_{k=1}^{\infty} : 0 < \lambda_1 < \lambda_2 \leq \lambda_3 \leq \dots \rightarrow \infty$ be the spectrum of the Dirichlet Laplacian L , $\{\phi_k\}_{k=1}^{\infty} : L\phi_k = \lambda_k\phi_k$ its eigenbasis in \mathcal{H} normalized by $(\phi_k, \phi_l) = \delta_{kl}$. The set of pairs

$$\Sigma_{\Omega} := \left\{ \lambda_k; \left. \frac{\partial\phi_k}{\partial\nu} \right|_{\Gamma} \right\}_{k=1}^{\infty}$$

is called the (Dirichlet) *spectral data* of the manifold Ω . The well-known fact is that these data determine the Weyl function and vice versa (see, e.g., [12]). Hence, Σ_{Ω} determines the minimal Laplacian L_0 up to unitary equivalence. However, such a determination can be realized not through M but in more explicit way.

Namely, let $U : \mathcal{H} \rightarrow \tilde{\mathcal{H}} := l_2$,

$$Uy = \tilde{y} := \{(y, \phi_k)\}_{k=1}^{\infty}$$

be the Fourier transform that diagonalizes L :

$$\tilde{L} := ULU^* = \text{diag} \{ \lambda_1, \lambda_2, \dots \}. \quad (2.24)$$

For any harmonic function $a \in \mathcal{A}$, its Fourier coefficients are

$$(a, \phi_k) = -\frac{1}{\lambda_k} \int_{\Gamma} a \frac{\partial \phi_k}{\partial \nu} d\Gamma$$

that can be verified by integration by parts. With regard to the latter, the spectral data Σ_{Ω} determine the image $\tilde{\mathcal{A}} := U\mathcal{A} \subset \tilde{\mathcal{H}}$ and its closure $\tilde{\mathcal{D}} = U\mathcal{D} = \text{clos } \tilde{\mathcal{A}}$, so that the determination

$$\Sigma_{\Omega} \Rightarrow \tilde{L}, \tilde{\mathcal{D}}$$

occurs. In the mean time, (2.4) implies

$$\tilde{L}_0 = U^* L_0 U = \tilde{L}|_{\tilde{L}^{-1}[\tilde{\mathcal{H}} \ominus \tilde{\mathcal{D}}]} \quad (2.25)$$

by isometry of U . Thus, \tilde{L}_0 is a unitary copy of L_0 constructed via the spectral data.

3 Applications

3.1 Inverse problems

In inverse problems (IP) for DSBC associated with manifolds, one needs to recover the manifold via its boundary inverse data ⁸. Namely,

IP 1: given for a fixed $T > \max_{x \in \Omega} \text{dist}(x, \Gamma)$ the response operator R^{2T} of the system (2.11)–(2.13), to recover the manifold Ω

IP 2: given the Weyl function M of the system (2.22)–(2.23), to recover the manifold Ω

IP 3: given the spectral data Σ_{Ω} , to recover the manifold Ω .

The problems are called *time-domain*, *frequency-domain*, and *spectral* respectively.

⁸In concrete applications (acoustics, geophysics, electrodynamics, etc), these data formalize the measurements implemented at the boundary.

Setting the goal to determine an unknown manifold from its boundary inverse data, we have to keep in mind the evident nonuniqueness of such a determination: all *isometric* manifolds with the mutual boundary have the same data. Therefore, the only reasonable understanding of "to recover" is to construct a manifold, which possesses the prescribed data [4].

As we saw, the common feature of the problems IP 1–3 is that their data determine the minimal Laplacian L_0 up to unitary equivalence. By this, each kind of data determines the wave spectrum Ω_{L_0} up to isometry. As will be shown, for a wide class of manifolds the relation $\Omega_{L_0} \stackrel{\text{isom}}{=} \Omega$ holds. Hence, for such manifolds, for solving the IPs it suffices to extract a unitary copy \tilde{L}_0 from the data, find its wave spectrum $\Omega_{\tilde{L}_0} \stackrel{\text{isom}}{=} \Omega_{L_0}$, and thus to get an isometric copy of Ω . It is the program for the rest of the paper.

3.2 Simple manifolds

Recall that we deal with a compact smooth Riemannian manifold Ω with the boundary Γ ; vol is the volume in Ω . Also, recall some definitions.

For a subset $A \subset \Omega$, denote by

$$\Omega^r[A] := \{x \in \Omega \mid \text{dist}(x, A) < r\}$$

the metric neighborhood of A of radius $r > 0$ and put $\Omega^0[A] := A$. Note that whatever A be, its neighborhood is an open set with the zero volume boundary:

$$\text{vol } \partial\Omega^r[A] = 0, \quad r > 0 \quad (3.1)$$

[8]. By A° we denote the set of its interior points: $x \in A^\circ$ if there is an $\varepsilon > 0$ such that $\Omega^\varepsilon[\{x\}] \subset A$. For a system $\alpha \subset 2^\Omega$, we define $\alpha^\circ := \{A^\circ \mid A \in \alpha\}$.

Let Y be a set, $\Xi \subset 2^Y$ a system of subsets. The system Ξ is said to be an *algebra* if

- $Y \in \Xi$
- $A, B \in \Xi$ implies $Y \setminus A, A \cap B \in \Xi$ (and hence $\emptyset, A \cup B \in \Xi$).

For a family $\alpha \subset 2^Y$, by $\Xi[\alpha]$ we denote the algebra generated by this family, i.e., the *minimal algebra* that contains α . As is known, $\Xi[\alpha]$ consists of the sets of the form $\cup_{n=1}^N \cap_{m=1}^M A_{nm}$, where A_{nm} or $Y \setminus A_{nm}$ belong to α (see, e.g., [6]).

Return to the manifold. The following is an universal process that associates with Ω a certain system $\alpha_\Gamma \subset 2^\Omega$ and that we refer to as a *Procedure*

1. The process is a consequent repetition of the same operation σ that acts as follows: for a given family $\alpha \subset 2^\Omega$,

1. constitute the algebra $\Xi[\alpha]$ and go to the system $\Xi^b[\alpha]$
2. construct the system of the neighborhoods

$$\sigma[\alpha] := \{\Omega^t[A] \mid A \in \Xi^b[\alpha], t \geq 0\}$$

that is the product of the operation σ .

Such an operation is of the following important feature. We say a set $A \subset \Omega$ to be *regular* (and write $A \in \mathcal{R} \subset 2^\Omega$) if $\text{vol } A > 0$ and $\text{vol } \partial A = 0$ holds. Note that \mathcal{R} is an algebra. The feature is that by the definition of the minimal algebra and property (3.1), the inclusion $\alpha \subset \mathcal{R}$ implies $\sigma[\alpha] \subset \mathcal{R}$. Also, note that the passage $\Xi[\alpha] \rightarrow \Xi^b[\alpha]$ removes the zero volume sets, which can appear in the algebra $\Xi[\alpha]$.

Now, we describe

Procedure 1:

Step 1 Take the family of boundary neighborhoods $\gamma := \{\Omega^t[\Gamma]\}_{t \geq 0} \subset 2^\Omega$ and construct the system $\sigma[\gamma]$

Step 2 Construct $\sigma^2[\gamma] := \sigma[\sigma[\gamma]]$

Step 3 Construct $\sigma^3[\gamma] := \sigma[\sigma[\sigma[\gamma]]]$

... ..

Final Step Constitute the system

$$\alpha_\Gamma := \bigcup_{j=1}^{\infty} \sigma^j[\gamma]$$

that is the end product of the Procedure 1. As is easy to see, the constructed system consists of regular sets, and is determined by the metric in Ω and the "shape" of its boundary Γ .

A system $\alpha \subset 2^\Omega$ is said to be a *net* if for any point $x \in \Omega$ there exists a sequence $\{\omega_j\}_{j=1}^{\infty} \subset \alpha$ such that $\text{vol } \omega_j > 0$, $\omega_1 \supset \omega_2 \supset \dots$, $\text{diam } \omega_j \rightarrow 0$, and $x \in \bigcap_{j \geq 1} \omega_j$.

We say the manifold Ω to be *simple*, if the system α_Γ is a net. The following is some comments on this definition.

The evident obstacle for a manifold to be simple is its symmetries⁹. For a ball $\Omega = \{x \in \mathbb{R}^n \mid |x| \leq 1\}$, the system α_Γ consists of the sets $\eta \times S^{n-1}$, where $\eta \subset [0, 1]$ is a sum of positive measure segments. Surely, such a system is not a net in the ball. A plane triangle is simple iff its legs are pair-wise nonequal. Sufficient and easily checkable conditions on the shape of $\Omega \subset \mathbb{R}^n$, which provide the simplicity, are proposed in [1]. They are also available for Riemannian manifolds and show that simplicity is a generic property: it can be reached by arbitrarily small smooth variations of the boundary Γ .

For an $A \subset \Omega$, define a distant function

$$d_A(x) := \text{dist}(x, A), \quad x \in \Omega.$$

By the well-known properties of the distance on a metric space, distant functions are continuous: $d_A \in C(\Omega)$. The latter space is a Banach algebra (with the sup-norm). Let $C_\Gamma(\Omega)$ be the (closed) subalgebra in $C(\Omega)$ generated by the family $\{d_A \mid A \in \alpha_\Gamma\}$. The following is the result, which in fact inspires the notion of simplicity.

Lemma 4 *If the manifold Ω is simple then the equality*

$$C_\Gamma(\Omega) = C(\Omega) \tag{3.2}$$

holds.

Proof For $x, x' \in \Omega$, $x \neq x'$, choose $\omega \in \alpha_\Gamma$ such that $x \in \omega$ and $x' \notin \overline{\omega}$, what is possible since the system α_Γ is a net. One has $0 = d_\omega(x)$ and $d_\omega(x') > 0$, so that $C_\Gamma(\Omega)$ distinguishes points of Ω . Therefore, by the Stone theorem, the sum $C_\Gamma(\Omega) \vee \{\text{constants}\}$ coincides with $C(\Omega)$. In the mean time, assuming $\{\text{constants}\} \not\subset C_\Gamma(\Omega)$, we have $C(\Omega) = C_\Gamma(\Omega) \dot{+} \{\text{constants}\}$ and conclude that $C_\Gamma(\Omega)$ is a maximal ideal in $C(\Omega)$. By the latter, there exists a (unique) point $x_0 \in \Omega$ such that all functions of $C_\Gamma(\Omega)$ vanish at x_0 (see, e.g., [11]). Evidently, it is not the case since α_Γ is a net. Hence, we arrive at (3.2). \square

3.3 Solving IPs

Here we prove the basic

⁹Presumably, any compact manifold with trivial symmetry group is simple, but it is a conjecture. Note that for noncompact manifolds this is not true.

Theorem 1 *Let Ω be a simple manifold, $L_0 = -\Delta|_{H_0^2(\Omega)}$ the minimal Laplacian, Ω_{L_0} its wave spectrum. There exists an isometry (of metric spaces) i that maps Ω_{L_0} onto Ω , the relation $i(\partial\Omega_{L_0}) = \Gamma$ being valid.*

Proof consists of the parts I–III.

I. System $\hat{\alpha}_\Gamma$

We say a system of subspaces $\hat{\Xi} \subset \text{Lat } \mathcal{H}$ to be an *algebra* if

- $\mathcal{H} \in \hat{\Xi}$
- $\mathcal{A}, \mathcal{B} \in \hat{\Xi}$ implies $\mathcal{H} \ominus \mathcal{A}, \mathcal{A} \cap \mathcal{B} \in \hat{\Xi}$ (and hence $\{0\}, \mathcal{A} \vee \mathcal{B} \in \hat{\Xi}$).

For a family $a \subset \text{Lat } \mathcal{H}$, by $\hat{\Xi}[a]$ we denote the algebra generated by this family, i.e., the *minimal algebra* that contains a . As is known, $\hat{\Xi}[a]$ consists of the subspaces of the form $\bigcup_{n=1}^N \bigvee_{m=1}^M \mathcal{A}_{nm}$, where \mathcal{A}_{mn} or $\mathcal{H} \ominus \mathcal{A}_{mn}$ belong to a .

Below we present an universal process that associates with Ω a certain system of subspaces $\hat{\alpha}_\Gamma \subset \text{Lat } \mathcal{H}$ and that we refer to as a *Procedure* $\hat{1}$. The process is a consequent repetition of the same operation $\hat{\sigma}$ that acts as follows: for a given family $a \subset \text{Lat } \mathcal{H}$,

1. constitute the algebra $\hat{\Xi}[a]$
2. construct the family

$$\hat{\sigma}[a] := \{E^r \mathcal{A} \mid \mathcal{A} \in \hat{\Xi}[a], r \geq 0\} \subset \text{Lat } \mathcal{H},$$

where $\{E^r\}_{r \geq 0} = E_L$ is the space extension determined by the Dirichlet Laplacian $L \supset L_0$.

The latter family is the product of the operation $\hat{\sigma}$. Such an operation is of the following important feature.

A subspace $\mathcal{A} \subset \mathcal{H}$ is called *regular* (we write $\mathcal{A} \in \hat{\mathcal{R}} \subset \text{Lat } \mathcal{H}$) if

$$\mathcal{A} = \{y \in \mathcal{H} \mid \text{supp } y \subset \overline{A}\} =: \mathcal{H}A$$

with an $A \in \mathcal{R}$; so, a regular subspace consists of functions supported on a regular set. Now, we invoke a fundamental property of the DSBC (2.11)–(2.13) known as a *local controllability*¹⁰, by which

¹⁰see [4], sec 2.2.3, eqn (2.21). This property is based upon the fundamental Holmgren-John-Tataru theorem on uniqueness of continuation of solutions to the wave equation (2.11) through a noncharacteristic surface: see [2] for detail.

- the embedding (2.14) is dense, i.e., in our current notation, we have $\text{clos}\mathcal{U}_+^t = \mathcal{H}\Omega^t[\Gamma]$, which shows that all $\text{clos}\mathcal{U}_+^t$ are regular subspaces. As a consequence, by (2.10) we conclude that the subspaces $\text{clos}\mathcal{U}^t$ are also regular:

$$\text{clos}\mathcal{U}^t = \mathcal{H}\Omega^t[\Gamma] \subset \hat{\mathcal{R}}, \quad t > 0 \quad (3.3)$$

- for any regular subspace $\mathcal{H}A$, the relation

$$E^t\mathcal{H}A = \mathcal{H}\Omega^t[A], \quad t \geq 0 \quad (3.4)$$

holds.

The above-announced feature of the operation $\hat{\sigma}$ is the following. By the definition of the minimal algebra and property (3.4), if $\hat{\sigma}$ is applied to a family a of regular subspaces, then the result $\hat{\sigma}[a]$ also consists of regular subspaces, i.e., $\hat{\alpha} \subset \hat{\mathcal{R}}$ implies $\hat{\sigma}[\hat{\alpha}] \subset \hat{\mathcal{R}}$.

Now, we describe

Procedure $\hat{1}$:

Step 1 Take the family of subspaces $\hat{\gamma} := \{\text{clos}\mathcal{U}^t\}_{t \geq 0} \subset \hat{\mathcal{R}}$ corresponding to the boundary neighbourhoods $\Omega^t[\Gamma]$ (see (3.3)) and construct the family $\hat{\sigma}[\hat{\gamma}]$

Step 2 Construct $\hat{\sigma}^2[\hat{\gamma}] := \hat{\sigma}[\hat{\sigma}[\hat{\gamma}]]$

Step 3 Construct $\hat{\sigma}^3[\hat{\gamma}] := \hat{\sigma}[\hat{\sigma}[\hat{\sigma}[\hat{\gamma}]]]$

... ..

Final Step Constitute the family

$$\hat{\alpha}_\Gamma := \bigcup_{j=1}^{\infty} \hat{\sigma}^j[\hat{\gamma}]$$

that is the end product of the Procedure $\hat{1}$. As is easy to see, the constructed family consists of regular subspaces: $\hat{\alpha}_\Gamma \subset \hat{\mathcal{R}}$.

The evident duality of the Procedures 1 and $\hat{1}$, which is intentionally emphasized by the notation, easily leads to the bijection

$$\mathcal{R} \supset \alpha_\Gamma \ni A \longleftrightarrow \mathcal{H}A \in \hat{\alpha}_\Gamma \subset \hat{\mathcal{R}}. \quad (3.5)$$

II. Eikonals Recall that for a linear set $\mathcal{A} \subset \mathcal{H}$, by $P_{\mathcal{A}}$ we denote the projection in \mathcal{H} onto $\text{clos}\mathcal{A}$.

By the bijection (3.5), each projection $P_{\mathcal{A}}$ with $\mathcal{A} \in \hat{\alpha}_{\Gamma}$ is of the form $P_{\mathcal{A}} = P_{\mathcal{H}A}$ with $A \in \alpha_{\Gamma}$. Hence, as an operator in $\mathcal{H} = L_2(\Omega)$, $P_{\mathcal{A}}$ multiplies functions by the indicator $\chi_A(\cdot)$ of the set A (cuts off functions on A), what we write as $P_{\mathcal{A}} = \chi_A$. By (3.4), the projection $P_{\mathcal{A}}^t = E^t P_{\mathcal{A}}$ cuts off functions on $\Omega^t[A]$, i.e., $P_{\mathcal{A}}^t = \chi_{\Omega^t[A]}$. As result, the operator

$$\tau_{P_{\mathcal{A}}} = \int_0^\infty t dP_{\mathcal{A}}^t = \int_0^\infty t d\chi_{\Omega^t[A]}$$

multiplies functions by the distant function:

$$(\tau_{P_{\mathcal{A}}} y)(x) = d_A(x) y(x), \quad x \in \Omega. \quad (3.6)$$

Let $\mathfrak{L}_{\infty}(\Omega)$ be the algebra of L_{∞} -multipliers, which is a von Neumann sub-algebra of $\mathfrak{B}(\mathcal{H})$. By $\mathfrak{C}_{\Gamma}(\Omega)$ and $\mathfrak{C}(\Omega)$ we denote the subalgebras of $C_{\Gamma}(\Omega)$ - and $C(\Omega)$ -multipliers respectively, both of them being closed w.r.t. the operator norm; so, we have $\mathfrak{C}_{\Gamma}(\Omega) \subset \mathfrak{C}(\Omega) \subset \mathfrak{L}_{\infty}(\Omega) \subset \mathfrak{B}(\mathcal{H})$. These subalgebras are commutative; also, as is well known, $\mathfrak{C}(\Omega)$ is weakly dense in $\mathfrak{L}_{\infty}(\Omega)$ [11]. Recall that \mathfrak{N}_{L_0} is defined in sec 1.3, whereas L_0 which we are dealing with, is the minimal Laplacian on Ω .

Lemma 5 *If the manifold Ω is simple then one has $\mathfrak{N}_{L_0} = \mathfrak{L}_{\infty}(\Omega)$.*

Proof (sketch) By construction of $\hat{\alpha}_{\Gamma}$, all projections $P_{\mathcal{A}}$ with $\mathcal{A} \in \hat{\alpha}_{\Gamma}$ belong to the algebra \mathfrak{N}_{L_0} . Whence, the eikonals of the form (3.6) belong to \mathfrak{N}_{L_0} , so that the embedding $\mathfrak{C}_{\Gamma}(\Omega) \subset \mathfrak{N}_{L_0}$ occurs.

By simplicity of Ω , one has $\mathfrak{C}_{\Gamma}(\Omega) = \mathfrak{C}(\Omega)$, what is just a form of writing the assertion of Lemma 4. So, we have $\mathfrak{C}(\Omega) \subset \mathfrak{N}_{L_0}$. The latter, by the above mentioned w-closeness of $\mathfrak{C}(\Omega)$ in $\mathfrak{L}_{\infty}(\Omega)$, implies $\mathfrak{L}_{\infty}(\Omega) \subset \mathfrak{N}_{L_0}$.

All projections belonging to $\mathfrak{L}_{\infty}(\Omega)$ are of the form $P = \chi_A$ with a Borel A [11]. Using the relevant generalization of the relation (3.4) on Borel positive volume sets, one can show that the algebra $\mathfrak{L}_{\infty}(\Omega)$ is invariant w.r.t. the extension E_L . Since \mathfrak{N}_{L_0} is a minimal E_L -invariant algebra, the embedding $\mathfrak{L}_{\infty}(\Omega) \subset \mathfrak{N}_{L_0}$ yields $\mathfrak{L}_{\infty}(\Omega) = \mathfrak{N}_{L_0}$. \square

Corollary 1 *The set of eikonals $\text{Eik } \mathfrak{N}_{L_0}$ consists of the operators of the form (3.6) with the Borel A 's.*

III. Wave spectrum We omit the proof of the following simple fact: the eikonal $\tau_{P_{\mathcal{A}}} = d_A$ is maximal iff A is a single point set, i.e., $A = \{x_0\}$ for

a $x_0 \in \Omega$. Such an eikonal is denoted by τ_{x_0} , so that we have

$$\Omega_{L_0} = \{\tau_{x_0} \mid x_0 \in \Omega\}$$

by the definition of a wave spectrum, whereas a map

$$i : \Omega_{L_0} \ni \tau_{x_0} \mapsto x_0 \in \Omega$$

is a bijection. Recall that the distance between eikonals $\tau', \tau'' \in \Omega_{L_0}$ is $\|\tau' - \tau''\|$ and show that i is an isometry.

Take $x', x'' \in \Omega$; the corresponding eikonals $\tau_{x'}, \tau_{x''} \in \Omega_{L_0}$ multiply functions by $d_{x'}$ and $d_{x''}$ respectively. For any $x \in \Omega$, by the triangular inequality one has $|d_{x'}(x) - d_{x''}(x)| \leq \text{dist}(x', x'')$; hence for $y \in \mathcal{H} = L_2(\Omega)$ one easily has $\|[\tau_{x'} - \tau_{x''}]y\| \leq \text{dist}(x', x'')\|y\|$ that implies

$$\|\tau_{x'} - \tau_{x''}\| \leq \text{dist}(x', x''). \quad (3.7)$$

Choose $\omega_j \subset \Omega$ such that $\omega_1 \supset \omega_2 \supset \dots$, $\text{diam } \omega_j \rightarrow 0$, and $x' = \bigcap_{j \geq 1} \omega_j$; then put $y_j = \|\chi_j\|^{-1} \chi_j$, where χ_j is the indicator of ω_j . As is easy to see, $\|\tau_{x'} y_j\| \rightarrow 0$ and $\|\tau_{x''} y_j\| \rightarrow \text{dist}(x', x'')$ holds as $j \rightarrow \infty$ that implies

$$\|[\tau_{x'} - \tau_{x''}]y_j\| \rightarrow \text{dist}(x', x''), \quad \|y_j\| = 1. \quad (3.8)$$

Comparing (3.7) with (3.8), we arrive at $\|\tau_{x'} - \tau_{x''}\| = \text{dist}(x', x'')$ and conclude that the map $\tau_{x_0} \mapsto x_0$ is an isometry.

Recall that the boundary of the wave spectrum is introduced in sec 1.3 by (1.11) and (1.12). In our case, in accordance with (3.3) the projections $P_{\mathcal{U}^t}$ cut off functions on the near-boundary subdomains $\Omega^t[\Gamma]$ and, correspondingly, the boundary eikonal $\tau^\partial = \int_0^\infty t dP_{\mathcal{U}^t}$ multiplies functions by $d_\Gamma := \text{dist}(\cdot, \Gamma)$. As is evident, the relation $d_{x_0} \geq d_\Gamma$ holds in Ω iff $x_0 \in \Gamma$. Equivalently, the eikonal $\tau_{x_0} \in \Omega_{L_0}$ satisfies $\tau_{x_0} \geq \tau^\partial$ iff $x_0 \in \Gamma$. This means that the isometry i maps the boundary $\partial\Omega_{L_0}$ of the wave spectrum onto the boundary Γ of the manifold.

Theorem 1 is proved. \square

Regarding non-simple manifolds, note the following. If the symmetry group of Ω is nontrivial then, presumably, Ω_{L_0} is isometric to the properly metricized set of the group orbits. Such a conjecture is supported by the following easily verifiable examples:

- for a ball $\Omega = \{x \in \mathbb{R}^n \mid |x| \leq r\}$, the spectrum Ω_{L_0} is isometric to the segment $[0, r] \subset \mathbb{R}$, whereas its boundary $\partial\Omega_{L_0}$ is identical to the endpoint $\{0\}$
- for an ellipses $\Omega = \{(x, y) \in \mathbb{R}^2 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} \leq 1\}$, Ω_{L_0} is isometric to its quarter $\Omega \cap \{(x, y) \mid x \geq 0, y \geq 0\}$, whereas $\partial\Omega_{L_0} \stackrel{\text{isom}}{=} \{(x, y) \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, x \geq 0, y \geq 0\}$
- let ω be a compact plane domain with the smooth boundary, which lies in the (open) upper half-plane \mathbb{R}_+^2 ; let Ω be a torus in \mathbb{R}^3 , which appears as result of rotation of ω around the x -axis. Then $\Omega_{L_0} \stackrel{\text{isom}}{=} \omega$ and $\partial\Omega_{L_0} \stackrel{\text{isom}}{=} \partial\omega$.

Actually, possible lack of simplicity is not an obstacle for solving the problems IP 1–3 because their data determine not only a copy of L_0 but substantial additional information relevant to the reconstruction of Ω . Roughly speaking, the matter is as follows. When we deal with each of these problems, the boundary Γ is given. By this, instead of the algebra $\mathfrak{N}[E_L, a] = \mathfrak{N}_{L_0}$ generated by the family $a = \{\text{clos } \mathcal{U}^t\}_{t \geq 0}$ of the sets reachable from *the whole* Γ (see (1.10) and (2.8)) we can invoke the wider algebra $\mathfrak{N}[E_L, a'] \supset \mathfrak{N}[E_L, a]$ generated by the much richer family $a' = \{\text{clos } \mathcal{U}_\sigma^t\}_{t \geq 0, \sigma \subset \Gamma} \supset a$ of the sets reachable from *any patch* $\sigma \subset \Gamma$ of positive measure¹¹. As result, although the equality $\mathfrak{N}_{L_0} = \mathfrak{L}_\infty(\Omega)$ may be violated by symmetries, the equality $\mathfrak{N}[E_L, a'] = \mathfrak{L}_\infty(\Omega)$ always holds, whereas the wave spectrum $\Omega_{\mathfrak{N}[E_L, a']}$ turns out to be isometric to Ω . The latter is the key fact, which makes the reconstruction possible: see [5] for detail.

3.4 Comments and remarks

A look at isospectrality As at the end of sec 2.3, let $\text{spec } L = \{\lambda_k\}_{k=1}^\infty$ be the spectrum of the Dirichlet Laplacian on Ω . The question: "Does $\text{spec } L$ determine Ω up to isometry?" is a version of the classical M.Kac's drum problem [9]. The negative answer is well known (see, e.g., [7]) but, as far as we know, the satisfactory description of the set of isospectral manifolds is not obtained yet. The following is some observations in concern with such a description.

¹¹ \mathcal{U}_σ^t consists of the solutions (waves) $u^f(t)$ produced by the boundary controls f supported on $\sigma \times [0, \infty)$

Assume that we deal with a simple Ω . In accordance with Theorem 1, such a manifold is determined by any unitary copy \tilde{L}_0 of the operator $L_0 \subset L$. If the spectrum of L is given, to get such a copy it suffices to possess the Fourier image $\tilde{\mathcal{D}} = U\mathcal{D}$ of the harmonic subspace in $\tilde{\mathcal{H}} = l_2$: see (2.25)¹². In the mean time, as is evident, if Ω and Ω' are isometric, then the corresponding images are identical: $\tilde{\mathcal{D}} = \tilde{\mathcal{D}}'$. Therefore, Ω and Ω' are isospectral but not isometric iff $\tilde{\mathcal{D}} \neq \tilde{\mathcal{D}}'$. In other words, the subspace $\tilde{\mathcal{D}}$ is a relevant "index" that differs the isospectral manifolds.

To be a candidate on the role of the harmonic functions image, which is admissible for the given $\tilde{L} = \text{diag} \{\lambda_1, \lambda_2, \dots\}$ (see (2.24)), a subspace $\tilde{\mathcal{D}} \subset l_2$ has to possess the following properties:

1. a lineal set $\mathcal{L}_{\tilde{\mathcal{D}}} := \tilde{L}^{-1} [l_2 \ominus \tilde{\mathcal{D}}]$ is dense in l_2 (see (2.25)), whereas replacement of $\tilde{\mathcal{D}}$ by any wider subspace $\tilde{\mathcal{D}}' \supset \tilde{\mathcal{D}}$ leads to the lack of density: $\text{clos } \mathcal{L}_{\tilde{\mathcal{D}}} \neq l_2$
2. extending an operator $\tilde{L}|_{\mathcal{L}_{\tilde{\mathcal{D}}}}$ by Friedrichs, one gets \tilde{L} (see (2.25)).

In the mean time, taking *any* subspace $\tilde{\mathcal{D}} \subset l_2$ provided 1 and 2¹³, one can construct a symmetric operator \tilde{L}_0 by (2.25) and then find its wave spectrum $\Omega_{\tilde{L}_0}$ as a candidate to be a drum. However, the open question is whether such a "drum" is human (is a manifold).

Model Once again, let Ω be simple. By Lemma 5, the algebra \mathfrak{N}_{L_0} is *cyclic*, i.e., possesses the elements $g \in \mathcal{H} = L_2(\Omega)$ such that

$$\text{clos } \{Pg \mid P \in \text{Proj } \mathfrak{N}_{L_0}\} = \mathcal{H}.$$

This enables one to realize elements of \mathcal{H} as functions on the wave spectrum of \mathfrak{N}_{L_0} by the following scheme:

- Fix a cyclic $g \in \mathcal{H}$ and endow Ω_{L_0} with a measure μ as follows. For a maximal eikonal $\tau = \int_0^\infty t dP_\tau^t \in \Omega_{L_0}$ and a ball $B_r[\tau] := \{\tau' \in \Omega_{L_0} \mid \|\tau' - \tau\| < r\}$ put

$$\mu(B_r[\tau]) := (P_\tau^r g, g)_{\mathcal{H}} \quad (3.9)$$

¹²It is the fact, which is exploited in [1]

¹³such subspaces do exist (M.M.Faddeev, private communication)

and then extend μ to the Borel subsets of Ω_{L_0} . As is easy to check, the equality

$$\mu(B_r[\tau]) = \int_{\Omega^r[x]} |g|^2 d\text{vol} \quad (3.10)$$

holds, where τ and x are related through the isometry $i : \Omega_{L_0} \rightarrow \Omega$ established by Theorem 1: $x = i(\tau)$.

So, we have a *model space* $\mathcal{H}_{\text{mod}} := L_{2,\mu}(\Omega_{L_0})$.

- For a $y \in \mathcal{H}$, define its image $Iy \in \mathcal{H}_{\text{mod}}$ by

$$(Iy)(\tau) := \lim_{r \rightarrow 0} \frac{(P_\tau^r y, g)_{\mathcal{H}}}{(P_\tau^r g, g)_{\mathcal{H}}}. \quad (3.11)$$

The relations (3.9) and (3.10) easily imply

$$(Iy)(\tau) = \lim_{r \rightarrow 0} \frac{\int_{\Omega^r[x]} y \bar{g} d\text{vol}}{\int_{\Omega^r[x]} |g|^2 d\text{vol}} = (g^{-1}y)(x) \quad (3.12)$$

and, hence, the image map I is a unitary operator from \mathcal{H} onto \mathcal{H}_{mod} .

An operator

$$L_0^{\text{mod}} := IL_0I^*$$

can be regarded as a functional model of the operator L_0 on its wave spectrum that we call a *wave model*. In the case under consideration, we have $L_0^{\text{mod}} = g^{-1}L_0g$, i.e., this model is just a gauge transform of the original. As such, it is a *local* operator:

$$\text{supp } L_0^{\text{mod}} w \subset \text{supp } w \quad (3.13)$$

holds for $w \in \text{Dom } L_0^{\text{mod}}$.

Conjectures

1. We suggest and hope that \mathcal{H}_{mod} and L_0^{mod} do exist for a wide class of symmetric semi-bounded operators L_0 , the locality property (3.13) being held. In contrast to the known models (see, e.g., [13]), this one has good chances to be of the real use for applications.

The principal point (and difficulty) is to attach the invariant meaning to the limit (3.11). Presumably, it can be done in the framework of the representation

$$\mathfrak{N}_{L_0} = \oplus \int_{\Omega_{L_0}} \mathfrak{N}_\tau d\mu(\tau) \quad (3.14)$$

in the form of a 1-st kind von Neumann algebra. Such a representation diagonalizes \mathfrak{N}_{L_0} and is expected to be valid for L_0 's coming from mathematical physics. As an encouraging example, the Maxwell system in electrodynamics can be mentioned: see [4]. Note that Maxwell's \mathfrak{N}_{L_0} is noncommutative.

2. A question of independent interest is whether any von Neumann algebra with space extension (in particular, \mathfrak{N}_{L_0}) is of the form (3.14). Also, in addition to the metric $\|\tau - \tau'\|$, it is reasonable to look for more subtle structures on Ω_{L_0} like tangent spaces, differentiable structure, etc.

3. One more attractive option is to construct a *wave model* of the abstract Green system satisfying Ryzhov's axiomatics, by the scheme

$$\{\mathcal{H}, \mathcal{G}; A, \Gamma_0, \Gamma_1\} \implies L_0 \implies \Omega_{L_0} \implies \{\mathcal{H}_{\text{mod}}, \mathcal{G}; A^{\text{mod}}, \Gamma_0^{\text{mod}}, \Gamma_1^{\text{mod}}\}$$

with $\mathcal{H}_{\text{mod}} = \oplus \int_{\Omega_{L_0}} \mathcal{H}_\tau d\mu(\tau)$ and a *local* A^{mod} . An intriguing point is that the boundary $\partial\Omega_{L_0}$ is well defined, so that there is a chance to realize $\Gamma_{0,1}^{\text{mod}}$ as the "true" trace operators. Such a model would provide a canonical realization of the original system, the realization being determined by its Weyl function M and, hence, relevant to inverse problems.

Unbounded case Return to the sec 1.1. If the assumption (1.1) is cancelled, the set $\text{Eik } \mathfrak{N}[E, a]$ is still well defined but unbounded. In particular, the case that *all* eikonals $\tau = \int_0^\infty t dP^t$ are unbounded operators is realized in applications: for instance, it holds if we deal with a noncompact simple manifold Ω . As result, even though in the mentioned example the relevant maximal eikonals do exist, in the general situation we have to correct the definition of the wave spectrum $\Omega_{\mathfrak{N}[E, a]}$. A possible way out is to deal with the *regularized* eikonals $\tau = \int_0^\infty \frac{t}{1+\alpha t} dP^t$ with a fixed $\alpha > 0$ and thus reduce the situation to the bounded case.

A bit of philosophy In applications, the external observer pursues the goal to recover a manifold Ω via measurements at the boundary Γ . The observer prospects Ω with waves u^f produced by boundary controls. These waves propagate into the manifold, interact with its inner structure and accumulate information about the latter. The result of interaction is also recorded at Γ . The observer has to extract the information from the recorded.

By the rule of game in IPs, the manifold itself is unreachable in principle. Therefore, the only thing the observer can hope for, is to construct from the measurements an *image* of Ω possibly resembling the original. By the same rule, the only admissible material for constructing is the waves u^f .

To be properly formalized, such a look at the problem needs two things:

- an object that codes exhausting information about Ω and, in the mean time, is determined by the measurements
- a mechanism that decodes this information.

Resuming our paper, the first is the minimal Laplacian L_0 , whereas to decode information is to determine its wave spectrum constructed from the waves u^f . It is Ω_{L_0} , which is a relevant image of Ω .

The current paper develops an algebraic trend in the BC-method [5], by which *to solve IPs is to find spectra of relevant algebras*. An attempt to use this philosophy for solving new problems would be quite reasonable. An encouraging fact is that in all above-mentioned unsolved IPs of anisotropic elasticity and electrodynamics, graphs with cycles, etc, the wave spectrum Ω_{L_0} does exist. However, to recognize how it looks like and verify (if true!) that $\Omega_{L_0} \stackrel{\text{isom}}{=} \Omega$ is difficult in view of very complicated structure of the corresponding reachable sets \mathcal{U}^t .

References

- [1] M.I.Belishev. On Kac's problem of the determination of shape of a domain from the spectrum of the Dirichlet problem. *Zapiski Nauch. Semin. POMI*, 173: 30–41, 1988 (in Russian); English translation: J. Sov. Math., v. 55 , no 3, 1991 .
- [2] M.I.Belishev. Boundary control in reconstruction of manifolds and metrics (the BC method). *Inverse Problems*, 13(5): R1–R45, 1997.
- [3] M.I.Belishev. Dynamical systems with boundary control: models and characterization of inverse data. *Inverse Problems*, 17 (2001), 659–682.
- [4] M.I.Belishev. Recent progress in the boundary control method. *Inverse Problems*, 23 (2007), no 5, R1–R67.
- [5] M.I.Belishev. Geometrization of Rings as a Method for Solving Inverse Problems. *Sobolev Spaces in Mathematics III. Applications in Mathematical Physics*, Ed. V.Isakov., Springer, 2008, 5–24.

- [6] M.S.Birman, M.Z.Solomyak. Spectral Theory of Self-Adjoint Operators in Hilbert Space. *D.Reidel Publishing Comp.*, 1987.
- [7] P.Buser, J.Conway, P.Doyle, K-D.Semmler. Some Planar Isospectral Domains. *International Math. Research Notices*, 1994, No 9, 391–400.
- [8] H.Federer. Curvature measures. *Trans. AMS*, vol. 93, 1959, 418–491.
- [9] M.Kac. Can one hear the shape of a drum. *Bull. Amer. Math. Monthly*, 73 (1966), 1–23.
- [10] A.N.Kotchubei. On extensions of symmetric operators and symmetric binary relations. *Mat.Zametki*, 17(1):41–48, 1975 (in Russian).
- [11] G.J.Murphy. C^* -Algebras and Operator Theory. *Academic Press, San Diego*, 1990.
- [12] V.Ryzhov. A General Boundary Value Problem and its Weyl Function. *Opuscula Math.*, 27 (2007), no. 2, 305–331.
- [13] A.V.Shtraus. Functional Models and Generalized Spectral Functions of Symmetric Operators. *St.-Petersburg Math. Journal*, 10(5): 1–76, 1998.